

Dedicated to Henry McKean

REALITY PROBLEMS IN THE SOLITON THEORY

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ABSTRACT. This is a survey article dedicated mostly to the theory of real regular “finite-gap” (algebro-geometrical) periodic and quasiperiodic Sine-Gordon solutions. Long period this theory remained unfinished and ineffective, and by that reason practically had no applications. Even for such simple physical quantity as “Topological Charge” no formulas existed expressing it through the “Inverse Spectral Data”. Few years ago the present authors solved this problem and made this theory effective. This article contains description of the history and recent achievements. It describes also the reality problems for several other fundamental soliton systems.

1. INTRODUCTION

The most powerful method for constructing explicit periodic and quasiperiodic solutions of soliton equations is based on the Finite-gap or Algebro-geometrical approach, developed by Novikov (1974), Dubrovin, Matveev, Its, Lax, McKean, Van-Moerbeke (1975) for $1 + 1$ systems, extended by Krichever (1976) for $2 + 1$ systems like KP (see [31], [20], [6], [24], [29] for extra information). Already in 1976 new ideas were formulated how to extend this approach to the $2 + 1$ systems associated with spectral theory of the 2D Schrodinger operator restricted to one energy level (see [26, 5]). These ideas were developed in 1980s by several people in the Moscow Novikov’s Seminar (see below). The “Spectral Data” characterizing the associated Lax-type operators consist of Riemann Surface (“Spectral Curve”) equipped by the selected set of points (“Divisor of Poles”, “Infinites”). In the finite gap case this Riemann surface has finite genus, and the number of selected point is also finite. The algebro-geometric approach in particular allows to write down explicit solutions in terms of the Riemann θ -functions.

In modern literature very often the problem is assumed “more or less” completely solved if such formulas are derived. However, in some cases this belief is too naive and does not correspond to the needs of real life. For example, it is necessary to select physically or geometrically relevant classes of solutions corresponding to the source problem (i.e.

solutions, satisfying some reality conditions, regular solutions, bounded solutions and so on). Is it easy or not?

To reach this goal, following problems should be solved.

- Problem 1. How to select solutions real for real (x, t) ?
- Problem 2. How to select real nonsingular solutions?
- Problem 3. How to select periodic solutions with given period (or quasiperiodic solutions with given group of quasiperiods)?

Remark. We call solution **non-singular**, if it is non-singular on the whole real Abel torus. It should remain non-singular under action of all (real) higher flows from the corresponding integrable hierarchy.

The generic x -direction is normally ergodic in the Abel torus, so this definition is equivalent to the standard one. However, for some specific values of constants of motion theoretically we may have solutions, which are regular in the standard sense, but blow-up under the action of the higher symmetries.

For some models like Korteweg-de Vries equation (KdV), defocusing Nonlinear Schrödinger Equation (NLS), Kadomtsev-Petviashvili 2 (KP2) equation selection of real and nonsingular solution is rather straightforward. But for many other models like focusing NLS, Sine-Gordon equation (SG), KP1, inverse scattering transformation for 2-D Schrödinger operator based at one energy the problem of selecting real solutions is rather difficult.

The theory of θ -functions is very complicated and ineffective. The complexity is hidden behind the simple notations in these formulas.

Our goal is to discuss in more details the Sine-Gordon equation:

$$(1) \quad u_{tt} - u_{xx} + \sin u(x, t) = 0.$$

In the light-cone coordinates

$$(2) \quad x = 2(\xi + \eta), \quad t = 2(\xi - \eta),$$

it has the form.

$$(3) \quad u_{\xi\eta} = 4 \sin u, \quad u = u(\xi, \eta).$$

According to our definition, the solution $u(x, t)$ is x -periodic with period T if $\exp\{iu\}$ is x -periodic with that period. For the function u we have

$$u(x + T, t) = u(x, t) + 2\pi n, n \in \mathbb{Z}.$$

We call the quantity n a **Topological Charge** corresponding the the period T .

We call the ratio n/T a **Density of Topological Charge**.

The Density of Topological Charge can be naturally extended to all real generic regular finite-gap (quasiperiodic) solutions. It is the most basic conservation law.

Problem: How to calculate topological charge of the real finite-gap solutions in terms of the spectral data.

Let us remind, that the Inverse Scattering (Spectral) Data for KdV and Sine-Gordon systems consist of a Riemann surface (Spectral Curve) Γ with finite genus equals to g and a collection of points (“divisor”) $D = \gamma_1 + \dots + \gamma_g$. (For NLS and some other systems number of poles may be different from genus).

In the case of KdV (or finite-gap periodic Schrödinger operator $L = -\partial_x^2 + u(x)$) this surface Γ is hyperelliptic. In the case of the Sine-Gordon equation the surface is also hyperelliptic, $\mu^2 = \lambda \prod_{i=1}^{2g} (\lambda - \lambda_i)$ with branching points $(0, \lambda_1, \dots, \lambda_{2g}, \infty)$. However the classes of admissible Riemann surfaces and divisors for KdV and Sine-Gordon are dramatically different (see below).

The θ -functional formulas for Sine-Gordon were obtained by Kozel, Kotlyarov and Its [22], [19] in 1976. The reality problem remained unsolved. Indeed, the class of admissible Riemann surfaces was found in these works (see [19]). The nonzero finite branching points $(\lambda_1, \dots, \lambda_{2g})$ can be either real negative $(\lambda_1, \dots, \lambda_{2k}) \in \mathbb{R}$ or complex conjugate with nonzero imaginary part $\lambda_{2k+1} = \bar{\lambda}_{2k+2}, \dots, \lambda_{2g-1} = \bar{\lambda}_{2g}$. However, no ideas were proposed where the poles are located on the Riemann Surface.

As it was realized in early 1980s by McKean [28], Dubrovin, Novikov and Natanzon [9], [7], Ercolani, Forest and McLaughlin [10],[11], this problem is nontrivial. By that reason periodic finite-gap Sine-Gordon theory long period had no applications.

An important idea how to describe position of poles for the real nonsingular solutions was in fact suggested by Cherednik in 1980 [4]. He was the first author who discovered (ineffectively) that for the given admissible real Riemann Surface there can be many different real Abel tori generating real nonsingular quasiperiodic solutions. Their number is equal to 2^k where $2k$ is the number of negative real branching points. All real finite-gap solutions are nonsingular for Sine-Gordon for the generic Riemann surface. His work was written in the abstract algebro-geometric form, and he never developed these ideas later. Extending this approach on the basis of “Algebro-Topological” ideas, Dubrovin and Novikov [9] presented an interesting idea how to calculate topological charge in terms of the “inverse spectral data”. However, as it

was pointed out by Novikov in 1984 [32], there was mistake in their arguments: the formula proposed in [9] was proved only for the small neighborhood of some very special solutions. The problem remained open till 2001. The complete solution (confirming Dubrovin-Novikov formula) was obtained by the authors in [13] as a development of the “Algebro-topological approach”, suggested in [9], see also [14], [15]. It is interesting that in the works [7] and later [10], these components were described as the real subtori in the Jacobian variety $J(\Gamma)$. However this “ θ -functional description” did not led yet to any formula for the topological charge. It does not require any specific basis of cycles. As we know now, good formula for the topological charge can be written in very specific basis only. We believe that using this basis of cycles one can deduce our formula from the θ -functional expression. It would be good to do that.

2. PHYSICALLY RELEVANT CLASSES OF SOLUTIONS FOR THE DIFFERENT SOLITON SYSTEMS

The Kortevveg-de Vries equation (KdV)

$$(4) \quad u_t + u_{xxx} - 6uu_x = 0, \quad u = u(x, t),$$

was originally derived in the water waves theory. As it was discovered in early 1960s (see introduction to the book [35]), it naturally appears as a first non-vanishing correction for the dispersive nonlinear systems if dissipation can be neglected. In these models only real non-singular solution are physically relevant.

Integration of KdV equation is based on the “Inverse Scattering Transform” for the 1-dimensional Schrödinger operator

$$(5) \quad L = -\partial_x^2 + u(x, t).$$

Selection of real KdV solutions is rather straightforward.

- (1) The spectral curve Γ $\mu^2 = R_{2g+1}(\lambda)$ should be real. It means, that $R_{2g+1}(\lambda) = \lambda^{2g+1} + \sum_{i=0}^{2g} p_i \lambda^i$ has real coefficients $p_i \in \mathbb{R}$, or equivalently, all roots are either real or form complex conjugate pairs. Therefore we have a holomorphic involution $\tau : (\lambda, \mu) \rightarrow (\bar{\lambda}, -\bar{\mu})$ on Γ .
- (2) The divisor D should be real with respect to τ : $\tau D = D$, or equivalently, the unordered set of points $\gamma_1, \dots, \gamma_g$ is invariant with respect to τ . Of course, τ may interchange some of them.

Real nonsingular KdV solutions correspond to the following special spectral data:

- (1) All branching points of λ_k of Γ are real and distinct. Assume, that $\lambda_1 < \lambda_2 < \dots < \lambda_{2g+1}$. Then τ has exactly $g + 1$ real ovals over the intervals $a_0 = (-\infty, \lambda_1]$, $a_1 = [\lambda_2, \lambda_3]$, \dots , $a_g = [\lambda_{2g}, \lambda_{2g+1}]$.
- (2) Each finite oval a_k , $1 \leq k \leq g$ contains exactly one divisor point $\gamma_k \in a_k$.

Remark. A real curve of genus g may have at most $g + 1$ real oval. Curves with $g + 1$ real ovals (maximal possible number) are called M -curves.

Generic finite-gap solutions are quasiperiodic with g incommensurable periods. How to select x -periodic solutions with prescribed period T ? Avoiding any use of Algebraic Geometry and Riemann Surfaces, nice approach to the characterization of the strictly x -periodic solution in terms of the so-called “quasimomentum map” was developed by Marchenko and Ostrovskii in 1975 [27]. This map was studied in the Quantum Solid State Physics Literature in 1959 (see [21]). It is well-defined in the upper half-plane outside of some vertical edges. Its analytical properties were effectively used in [27]. For example the approximation of x -periodic solution (potential) by the finite-gap ones periodic with the same period, was proved. Another approach based on the so-called isoperiodic deformations of finite-gap potentials was developed by Grinevich and Schmidt in 1995 [17]. In the KdV case the isoperiodic deformations can be interpreted as the so-called “Loewner equations” for the corresponding conformal map. Let us point out, that there exists a big literature, dedicated to the KdV solutions with real poles (rational solutions, singular trigonometric and elliptic solutions) – see [1] where these ideas were started. These solutions are very important from the mathematical point of view: for example, the dynamics of poles satisfies to the equations of the rational and elliptic Moser-Calogero models respectively. However, they are related neither to nonlinear wave problems nor to the spectral theory of the corresponding Schrödinger operators. So we do not discuss this literature in the present survey article.

The modified Korteweg-de Vries equation has the form:

$$(6) \quad v_t + v_{xxx} - 6v^2v_x = 0, \quad v = v(x, t).$$

It is connected with KdV by the Miura transformation:

$$(7) \quad u(x, t) = v_x(x, t) + v^2(x, t).$$

The real non-singular solutions are physically relevant.

The “complex” Non-linear Schrödinger equation (NLS) is a system of equations for the pair of independent complex functions $q = q(x, t)$,

$r = r(x, t)$:

$$(8) \quad \begin{cases} iq_t + q_{xx} + 2q^2r = 0 \\ -ir_t + r_{xx} + 2r^2q = 0 \end{cases}$$

This system has 2 natural real reductions: defocusing NLS: $r(x, y) = -\overline{q(x, y)}$

$$(9) \quad iq_t + q_{xx} - 2|q|^2q = 0,$$

and self-focusing NLS $r(x, y) = \overline{q(x, y)}$

$$(10) \quad iq_t + q_{xx} + 2|q|^2q = 0.$$

These equation describes nonlinear media with dispersion relations depending on the square of the wave amplitude (see [35]). Among the todays applications of NLS is the theory of light propagation in the fiber optics. The sign $+$ or $-$ is determined by the dispersion relation, and the qualitative behaviour critically depends on it. From the mathematical point of view, the defocusing NLS system is much simpler because the linear Lax operator is self-adjoint. The focusing NLS is much more complicated. In both cases physical applications requires regular solutions.

The complex NLS spectral data are following: A hyperelliptic Riemann surface Γ with $2g + 2$ finite branching points $\lambda_1, \dots, \lambda_{2g+2}$ and $g + 1$ divisor points $D = \gamma_1 + \dots + \gamma_{g+1}$. In contrast with the KdV case, there is no branching at ∞ .

Solutions of the defocusing NLS correspond to the following spectral data:

- (1) Γ is real, i.e. the polynomial $R_{2g+2} = \prod_{k=1}^{2g+2} (\lambda - \lambda_k)$ has real coefficients. Γ is defined by $\mu^2 = R(\lambda)$. The anti-holomorphic involution on Γ is defined by the map $\tau : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ where $(\lambda, \mu) \rightarrow (\bar{\lambda}, -\bar{\mu})$.
- (2) The divisor D is real with respect to τ : $\tau D = D$.

Selection of regular solutions is also very similar to the KdV case

- (1) All branching points of Γ are real. Therefore Γ has $g + 1$ real ovals over the intervals $[\lambda_{2k-1}, \lambda_{2k}]$, $k = 1, \dots, g + 1$, i.e. Γ is an M -curve.
- (2) There is exactly one divisor point at each real oval.

Selection of x -periodic solutions is completely analogous to the KdV case.

Let us describe the data generating real solutions of the self-focusing NLS equations for regular spectral curves. By the Cherednik theorem

[4], these solutions are automatically non-singular. Solutions, corresponding to singular spectral curves can be obtained as proper degenerations. In contrast with the defocusing case, singular curves may generate regular x -quasiperiodic solutions.

- (1) Γ is a real hyperelliptic surface of genus g with $2g+2$ $\lambda_1 < \lambda_2 < \dots < \lambda_{2g+2}$ finite branching points. There are no branching points on the real line, so they form complex conjugate pairs. The antiholomorphic involution τ acts on the λ -plane as $\tau\lambda = \bar{\lambda}$. The points in Γ lying over the real line are invariant with respect to τ . Equivalently, $\tau : (\lambda, \mu) \rightarrow (\bar{\lambda}, \bar{\mu})$.
- (2) There exists a meromorphic differential Ω such, that:
 - $\Omega = (1 + o(1))d\lambda$ at the infinite points of Γ .
 - Ω is regular outside infinity. Therefore it has exactly $2g+2$ zeroes.
 - Let $D = \gamma_1 + \dots + \gamma_{g+1}$. Then the divisor of zeroes of Ω is $D + \tau D$. Therefore, $D + \tau D = 2\infty_1 + 2\infty_2 - K$.

The Sine-Gordon equation in the light-cone variables was derived in the end of the 19-th century. It describes immersions of the negative curvature surfaces into \mathbb{R}^3 . Assume, that an asymptotic coordinate system is chosen (a coordinate system such, that coordinate lines have zero normal curvature). The angle between the coordinate lines satisfy (3). It means, that only real regular solutions such that $u(x, t) \neq 0 \pmod{\pi}$, are relevant.

The Sine-Gordon equation describes also dynamics of the Josephson junctions. In this model $u(x, t)$ is the phase difference between the contacts, therefore the real non-singular solutions are relevant. However, according to the leading experts in the Superconductivity Theory, the problem always requires boundary problem, so we have to consider either the finite interval or the half-line.

The elliptic Sinh-Gordon equation

$$(11) \quad u_{xx} + u_{yy} + 4H \sinh u = 0.$$

describes the constant mean curvature surfaces with genus equal to one, outside umbilic points (see review [2]). The constant mean curvature tori has no umbilic points, therefore real nonsingular solutions should be selected. In contrast with soliton equations, all real smooth double-periodic solutions are automatically finite-gap here: [18], [36]. It follows from the following observations by Hitchin [18]: all isospectral flows from the corresponding hierarchy are zero eigenfunctions of the linearized problem. But the linearized system is the 2-dimensional (elliptic) Schrödinger operator, and it may have only finite-dimensional

space of double-periodic zero eigenfunctions. It means, that the hierarchy contains only finite number of linearly independent flows at this point. As a corollary the spectral curve has finite genus. A further development of this idea was used by Novikov and Veselov in the paper [34]. It was shown, that all periodic chains of Laplace transformations consisting of the 2-dimensional double-periodic Schrödinger operators with regular coefficients are algebro-geometric (2D analogs of “finite-gap” operators).

The Boussinesq equation

$$(12) \quad \begin{cases} u_t = \eta_x \\ \eta_t = -\frac{1}{3}u_{xxx} + \frac{4}{3}uu_x. \end{cases}$$

is used for describing the water waves. For physical applications it is necessary to select real non-singular solutions. We would like to point out, that the problem of selecting such solutions in terms of the finite-gap data remains open.

The Kadomtsev–Petviashvili (KP) equation

$$(13) \quad (u_t + u_{xxx} - 6uu_x)_x + 3\alpha^2 u_{yy} = 0, \quad u = u(x, y, t), \quad \alpha^2 \in \mathbb{R}.$$

The auxiliary linear operator for KP has the form

$$(14) \quad L = \alpha \partial_y - \partial_x^2 + u(x, y, t).$$

If α is imaginary, we have the so-called KP1 equation, and L is the one-dimensional non-stationary Schrödinger operator. If α is real, L is the parabolic operator. In both cases the real non-singular solutions are physically relevant only. The necessary and sufficient conditions for the finite-gap spectral data selecting the real non-singular solutions were found by Dubrovin and Natanzon [8].

Real non-singular solutions of the KP-2 equation correspond to the following geometry:

- (1) Γ is a algebraic surface of genus g with a marked point and an antiholomorphic involution τ such, that the marked point is invariant under the action of τ . The marked point is the essential singularity of the wave function.
- (2) τ has exactly $g + 1$ fixed oval, i.e. Γ in an M -curve with respect to τ . Denote the oval containing the essential singularity by a_0 and the other ovals by a_n , $n = 1, \dots, g$.
- (3) Each oval a_n , $n \neq 0$ contains exactly one divisor point.

In the case of the Kadomtsev–Petviashvili 1 equation the reality constraints on the spectral curve are exactly the same as in the KP 2 case, but the divisor D has a completely different description: There exists a meromorphic differential Ω with exactly one second-order pole located

at the marked point such, that the divisor of zeroes of Ω is exactly $D + \tau D$. Equivalently, $D + \tau D = 2\infty - K$, where ∞ denotes the marked point. Regular real solutions are generated by the data with the following extra constraint:

The pair (Γ, τ) is of separating type, i.e. after removing all real ovals Γ splits into 2 components.

An important example of “solvable” inverse spectral transform is the one-energy problem for the two-dimensional Schrödinger operator started in the works [26, 5].

$$(15) \quad L = -\partial_x^2 - \partial_y^2 + u(x, y),$$

It is well-known, that the full set of the scattering data for multidimensional Schrödinger operators $n > 1$ is overdetermined. A lot of people studied this problem. We even don’t quote this literature. However, the case $n = 2$ turned out to be very specific. In 1976 Manakov, Dubrovin, Krichever and Novikov [26, 5] started completely new approach for this specific case: They started the Inverse Scattering theory and corresponding Soliton Theory associated with one selected energy level. A lot of work was done later in this subject later (see [33, 37, 16] and review [12] for additional references). In particular, in the first work [5] they defined the natural analogs of finite-gap potentials for the 2-dimensional problem as the potentials, “finite-gap at one energy”. Let $u(x, y)$ be double-periodic. Denote the dispersion relation by $\epsilon_j(k_x, k_y)$. The Fermi-curve at the energy level E_0 is defined by:

$$(16) \quad \epsilon_j(k_x, k_y) = E_0$$

Denote the complex continuation of the Fermi-curve by Γ . The Potential $u(x, y)$ is called **finite-gap at one energy**, if Γ has finite genus.

For generic spectral data the operators constructed in [5] have generically a non-zero magnetic field, i.e. they have some extra first-order terms:

$$(17) \quad L = -\partial_x^2 - \partial_y^2 + A_1(x, y)\partial_x + A_2(x, y)\partial_y + u(x, y),$$

It might happen that $H(x, y) \neq 0$, where $H(x, y) = \partial_x A_2(x, y) - \partial_y A_1(x, y)$. For physical applications it is important to select the case of “potential operators” $A_1(x, y) = A_2(x, y) = 0$ with real potential $u(x, y)$. Sufficient conditions on the spectral data leading to the potential operators were found in 1984 by Novikov and Veselov in [33]. For double periodic potentials the existence of such form is necessary. It follows from the direct spectral theory, developed by Krichever [23].

For the generic regular quasiperiodic potentials “finite-gap for one energy level”, this problem remains open. Selection of real potentials here is simple.

How to select the class of regular potentials in terms of algebro-geometrical spectral data? There is no complete solution to this Problem. It was shown in [33], that if the spectral curve is the so-called M -curve, then the potential $u(x, y)$ is regular, and the operator L is strictly positive (the selected energy level lies below the ground state). An alternative proof of the last statement was obtained by the authors in [16]. The complete characterization of the data generating strictly positive operators (with real regular potentials) was “more or less” clarified but some special features remain unproved rigorously.

If the selected energy level is located above the ground state, the topology of the spectral curve Γ become more complicated. Many classes of spectral data generating real non-singular solutions were found by Natanzon (see review [30]), but the classification is not complete till now.

3. SINE-GORDON EQUATION.

Connections between the Sine-Gordon equation and the inverse scattering method were first established by G.Lamb in 1971 [25]. The modern approach developed by Ablowitz, Kaup, Newell and Segur in 1974 [3] is based on the following zero-curvature representation:

$$(18) \quad \Psi_x = \frac{1}{4}(U + V)\Psi, \quad \Psi_t = \frac{1}{4}(U - V)\Psi,$$

where

$$(19) \quad U = U(\lambda, x, t) = \begin{bmatrix} i(u_x + u_t) & 1 \\ -\lambda & -i(u_x + u_t) \end{bmatrix},$$

$$(20) \quad V = V(\lambda, x, t) = \begin{bmatrix} 0 & -\frac{1}{\lambda}e^{iu} \\ e^{-iu} & 0 \end{bmatrix}.$$

As we mentioned above, the finite-gap “spectral data” consist of

- (1) A hyperelliptic Riemann surface Γ with $2g + 2$ branching points

$$(0, \lambda_1, \dots, \lambda_{2g}, \infty): \mu^2 = \lambda \prod_{i=1}^{2g} (\lambda - \lambda_i)$$

- (2) The divisor (a collection of points) $D = \gamma_1 + \dots + \gamma_g$ in Γ .

In our text we always assume, that the spectral curve Γ is **generic**, i.e. all branching points are distinct.

Construction of the complex Sine-Gordon solutions is based on the following standard Lemma:

Lemma 1. *For generic data Γ, D there exists a unique two-component vector-function $\Psi(\gamma, x, t)$ (the “Baker-Akhiezer” functions) such that*

- (1) *For fixed (x, t) the function $\Psi(\gamma, x, t)$ is meromorphic in the variable $\gamma \in \Gamma$ outside the points $0, \infty$ and has at most 1-st order poles at the divisor points $\gamma_k, k = 1, \dots, g$.*
- (2) *$\Psi(\gamma, x, t)$ has essential singularities at the points $0, \infty$ with the following asymptotic:*

$$(21) \quad \Psi(\gamma, x, t) = \left(\begin{array}{c} 1 + o(1) \\ i\sqrt{\lambda} + O(1) \end{array} \right) e^{\frac{i\sqrt{\lambda}}{4}(x+t)} \quad \text{as } \lambda \rightarrow \infty,$$

$$(22) \quad \Psi(\gamma, x, t) = \left(\begin{array}{c} \phi_1(x, t) + o(1) \\ i\sqrt{\lambda}\phi_2(x, t) + O(\lambda) \end{array} \right) e^{-\frac{i}{4\sqrt{\lambda}}(x-t)} \quad \text{as } \lambda \rightarrow 0,$$

with some $\phi_1(x, t), \phi_2(x, t)$.

The Sine-Gordon potential $u(x, t)$ is defined by:

$$(23) \quad u(x, t) = i \ln \frac{\phi_2(x, t)}{\phi_1(x, t)}.$$

Denote by $\lambda_k(x, t)$ the projections of the zeroes of the first component of $\Psi(\gamma, x, t)$ to the λ -plane. Then

$$(24) \quad e^{iu(x, t)} = \prod_{j=0}^g (-\lambda_j(x, t)) \left/ \sqrt{\prod_{j=1}^{2g} E_j} \right.$$

Remark. To be more precise, the formulas (18-24) define simultaneously a pair of Sine-Gordon solutions $u_1(x, t), u_2(x, t)$, depending on the choice of the branch $1/\sqrt{\lambda}$ near the point $\lambda = 0$. They are connected by the following relation $u_2(x, t) = u_1(t, x) + \pi$. In the real case it is possible to fix a canonical branch by making the analytical continuation along the real line. This rule is unstable in the following sense: if we add a pair of complex conjugate branching points which are very close to the positive half-line (or, equivalently, open a resonant point), it is a small transformation in terms of the spectral data, but it exchanges u_1 with u_2 .

The real Sine-Gordon solutions (by Cherednik’s lemma they are automatically regular [4]) correspond to the following data:

- (1) Γ is real, i.e. the branching points of Γ are either real, or form complex conjugate pairs. Therefore we have an antiholomorphic involutions $\tau : (\lambda, \mu) \rightarrow (\bar{\lambda}, \bar{\mu})$. Denote the number of real finite branching points by $2k + 1$.

- (2) All real branching points lie in the negative half-line $\lambda \leq 0$. It is convenient to use following enumeration for the branching points different from 0 and ∞ : $0 > \lambda_1 > \lambda_2 > \dots > \lambda_{2k}$, $\lambda_{2k+1} = \bar{\lambda}_{2k+2}, \dots, \lambda_{2g-1} = \bar{\lambda}_{2g}$.
- (3) There exists a meromorphic differential Ω (Cherednik differential) with first order poles at 0, ∞ , holomorphic on $\Gamma \setminus \{0, \infty\}$ with the zeroes at the points $\gamma_1, \dots, \gamma_g, \tau\gamma_1, \dots, \tau\gamma_g$ (or, equivalently the divisor D satisfy the relation $D + \tau D = 0 + \infty - K$).

As it was shown in [4], the variety of all real potentials corresponding to the given spectral curve Γ consists of 2^k connected components. A characterization of these components in terms of the Abel tori was obtained in [7] but this technique did not led to the calculation of topological charge through the inverse spectral data.

Our calculation of the topological charge for the finite-gap Sine-Gordon solutions is based on the following effective description of these components (see [13], [14], [15]):

Any meromorphic differential with first-order pole at ∞ can be written as:

$$(25) \quad \Omega = c \left(1 - \frac{\lambda P_{g-1}(\lambda)}{R(\lambda)^{1/2}} \right) \frac{d\lambda}{2\lambda},$$

where $P_{g-1}(\lambda)$ is a polynomial of degree at most $g-1$. It is also natural to put $c = 1$. In case of the Cherednik differentials the set of zeroes is invariant with respect to τ . Therefore all coefficients of the polynomial $P_{g-1}(\lambda)$ are real.

Assume, that we take an arbitrary real polynomial $P_{g-1}(\lambda)$. Is it possible to construct a real Sine-Gordon solution corresponding to it? The necessary and sufficient condition is the following: **the zeroes of Ω can be divided into two groups $\{\gamma_1, \dots, \gamma_g\}$ and $\{\gamma_{g+1}, \dots, \gamma_{2g}\}$ such, that $\tau\gamma_k = \gamma_{k+g}$, $k = 1, \dots, g$.** Equivalently, a polynomial $P_{g-1}(\lambda)$ generates real SG solutions if and only if all real root of Ω have even multiplicity. In generic situation (all roots form distinct complex conjugate pairs) each polynomial $P_{g-1}(\lambda)$ generates 2^g different solutions. To choose one of them one has to say, which point to choose in each complex conjugate pair belonging to D (the second one belongs to τD). In degenerate cases (i.e. if there are real roots) the number of choices is smaller. All these solutions associated with a given $P_{g-1}(\lambda)$ belong to the same real Abel torus.

Definition. A polynomial $P_{g-1}(\lambda)$ (and the corresponding differential Ω) are called **admissible** if all real roots of Ω have even multiplicity.

Admissible polynomials $P_{g-1}(\lambda)$ can be characterized in the following geometrical way:

Let us draw the graph of the functions

$$(26) \quad f_{\pm}(\lambda) = \pm \frac{\sqrt{R(\lambda)}}{\lambda},$$

and fill in the following domains by the black color:

$$(27) \quad \begin{aligned} \lambda < 0, y^2 &< \frac{R(\lambda)}{\lambda^2}, \\ \lambda > 0, y^2 &> \frac{R(\lambda)}{\lambda^2}. \end{aligned}$$

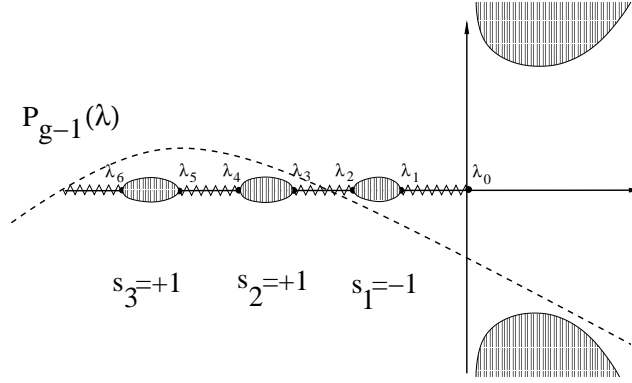


Fig 1.

Lemma 2. *The polynomial $P_{g-1}(\lambda)$ is admissible if and only if the graph of $P_{g-1}(\lambda)$ has no parts inside the black open domains.*

If the graph does not touch these domains, we have no real divisor points. Real divisor points correspond to the case, when the graph touches one of these domains but does not cross the boundary.

Each pair $\lambda_{2j-1}, \lambda_{2j}$ is connected by a black “island”. The graph of admissible $P_{g-1}(\lambda)$ should go above or below this island, therefore at all intervals $[\lambda_{2j}, \lambda_{2j-1}]$, $j \leq k$ $P_{g-1}(\lambda) \neq 0$. Let us associate with an admissible polynomial $P_{g-1}(\lambda)$ a collection of numbers s_j , $j = 1, \dots, k$ by the following rule: $s_j = 1$ if the graph of $P_{g-1}(\lambda)$ is positive in the interval $[\lambda_{2j}, \lambda_{2j-1}]$ and $s_j = -1$ otherwise. Let us call the set s_j **topological type of the real solution**. We have exactly 2^k possible topological types. Elementary analytic estimates (see [14]) show, that all these components are non-empty. Each connected component is a real Abel torus, and the x -dynamics defines a straight line in this torus. To calculate the density of the topological charge it is sufficient to know the direction of this line and the charges along the basic cycles. It follows from a simple analytic lemma:

Lemma 3. *Let $u(\vec{X})$, $X \in \mathbb{R}^n$ be a smooth function in \mathbb{R}^n such, that $\exp(iu(\vec{X}))$ is single-valued on the torus $\mathbb{R}^n/\mathbb{Z}^n$. Equivalently it means,*

that $\exp(iu(\vec{X} + \vec{N})) = \exp(iu(\vec{X}))$ for any integer vector \vec{N} , and

$$(28) \quad u(X^1, X^2, \dots, X^k + 1, \dots, X^n) - u(X^1, X^2, \dots, X^k, \dots, X^n) = 2\pi n_k.$$

The numbers n_k are called **the topological charges along the basic cycles** $\mathfrak{A}_k, k = 1, \dots, n$. Denote by $u(x)$ restriction of $u(\vec{X})$ to the strait line $\vec{X} = \vec{X}_0 + x \cdot \vec{v}$, $\vec{v} = (v^1, v^2, \dots, v^n)$. Then the density of topological charge $\bar{n} = \lim_{T \rightarrow \infty} [u(x + T) - u(x)]/2\pi T$ is well-defined; it does not depend on the point \vec{X}_0 and:

$$(29) \quad \bar{n} = \sum_{k=1}^n n_k v^k.$$

The calculation of the direction vector for the x -dynamics is absolutely standard (see, for example [10]). Denote by ω^l the canonical basis of holomorphic differentials on Γ :

$$(30) \quad \omega^l = i \frac{\sum_{j=0}^{g-1} D_j^k \lambda^j}{\sqrt{R(\lambda)}} d\lambda, \quad D_j^k \in \mathbb{R}$$

Then for the components of the x -direction vector we have:

$$(31) \quad U_k = \frac{1}{2} \left(D_{g-1}^k + D_0^k / \sqrt{\prod_{j=1}^{2g} E_j} \right).$$

To obtain a simple expression for the basic charges it is critical to use a proper basis of cycles in Γ . In [13]-[15]) the authors used the following basis, first suggested in [9]:

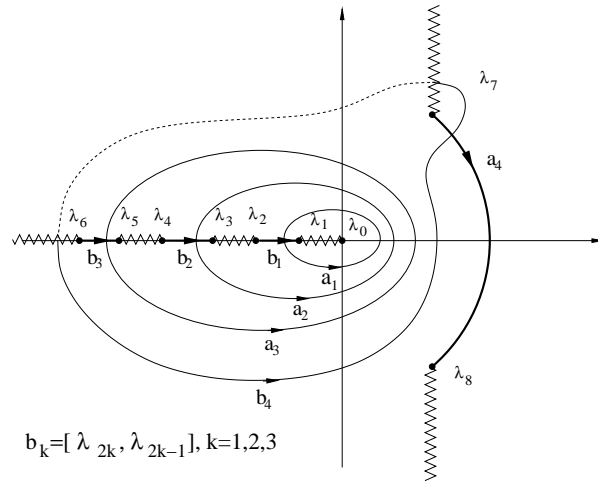


Fig 2.

Here the cycles a_j , $j = 1, \dots, k$ are ovals on the upper sheet of Γ , containing inside the points $\lambda_0 = 0, \lambda_1, \lambda_2, \dots, \lambda_{2j-1}$. The cycle b_j , $1 \leq j \leq k$ lies over the interval $[\lambda_{2j}, \lambda_{2j-1}]$. The cycles a_j , $j = k+1, \dots, g$ lie over pathes connecting the pairs λ_{2j-1} and λ_{2j} . We assume that these cycles do not intersect each other, and the cycles a_j , $j = k+1, \dots, g$ do not intersect the negative semi-line. The cuts are shown by the zigzag lines. The upper sheet contains the semi-line $\lambda > 0, \mu > 0$.

Consider a basic cycle \mathfrak{A}_j on the real component of Jacoby torus, represented by the closed curve. The image of this cycle in Γ under the inverse Abel map is a closed oriented curve C_j , formed by the motion of the corresponding divisor points (it may have several connected components). The motion of an individual divisor point does not have to be periodic, after going along the cycle we may obtain a permutation of the divisor points. The curve C_j is homological to the cycle $a_j \in H_1(\Gamma, \mathbb{Z})$. It follows from (24) that the topological charge n_j along the cycle \mathfrak{A}_k equals to the winding number of the curve C_j with respect to the point 0. Equivalently

$$(32) \quad n_j = \tilde{C}_j \circ \mathbb{R}_-,$$

where \circ denote the intersection number, \tilde{C}_j denotes the projection of C_j to the λ -plane, \mathbb{R}_- is negative semi-line with the standard orientation.

For each point of \mathfrak{A}_j the corresponding divisor $\gamma_1, \dots, \gamma_g$ is admissible. From the characterization of admissible divisors obtained above it is easy to show, that the curve C_j does not touch the closed segments on the real line $[-\infty, \lambda_{2m}], \dots, [\lambda_3, \lambda_2], [\lambda_1, 0]$. Therefore any time the curve C_j crosses the negative semi-line, it intersects one of the basic cycles b_j , $j = 1, \dots, k$.

Unfortunately this information is not sufficient to calculate the basic charge, because the orientation of the cycles b_j coincides with the orientation of the negative semi-line at one sheet and they are opposite at the other one. For example at the Fig. 3 we see two different realizations of the cycle a_1 representing different topological types. a_1 is drawn at the upper sheet and a'_1 is drawn at the lower one. We have $a_1 \circ b_1 = a'_1 \circ b_1 = 1$, but $\tilde{a}_1 \circ \mathbb{R}_- = 1$, $\tilde{a}'_1 \circ \mathbb{R}_- = -1$, therefore $n_1 = 1$ and $n_1 = -1$ for these cycles respectively.

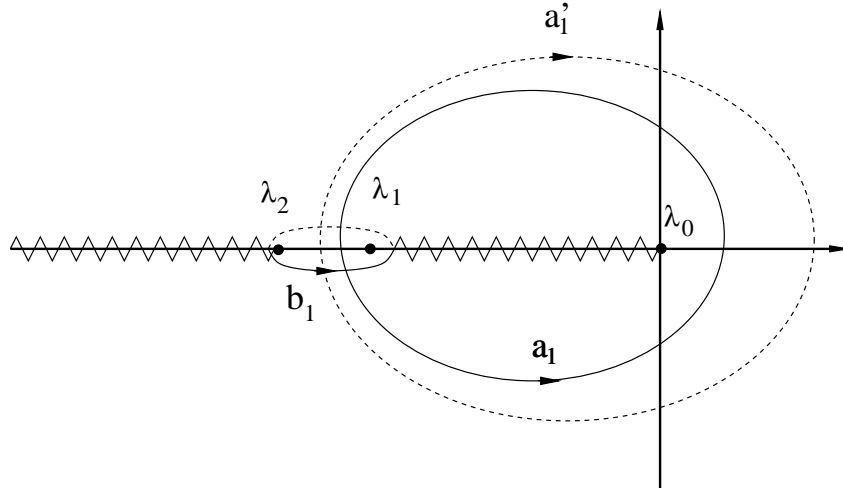


Fig 3.

Fortunately, the topological type contains the information at which sheet the intersection takes place. Namely we have:

Lemma 4. *Assume, that the cycles C_j intersects the negative semi-line at the interval $(\lambda_{2l}, \lambda_{2l-1})$. Then orientations of b_l and \mathbb{R}_- coincide in the intersection point if $(-1)^{l-1}s_l > 0$ and are opposite if $(-1)^{l-1}s_l < 0$.*

Combining all these results we obtain the final formula:

Theorem 1. *The density of the topological charge for a real Sine-Gordon solution is given by*

$$(33) \quad \bar{n} = \sum_{j=1}^k (-1)^{j-1} s_j U_j,$$

where the vector U_j is defined by (31).

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